Violation of Rotational Invariance of Local Realistic Models with Two Settings

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We have considered a two-particle Bell experiment to visualize the conflict between rotational invariance of physical laws and a specific local realistic theory. The experiment is reproducible by using a local realistic theory obtained in a two-setting Bell experiment. The generalized Bell inequality [J. Phys. A: Math. Theor. **40**, 13101 (2007)], which is derived under the assumption that there exists a rotationally invariant local realistic theory, turns out to disprove such a local realistic model existing with a two-setting experiment. This implies that such a model is not rotationally invariant and should, therefore, be ruled out in some situations.

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I. INTRODUCTION

Local and realistic theories assume that physical properties exist irrespective of whether they are measured and that the result of measurement pertaining to one system is independent of any other measurement simultaneously performed on a different system at a distance. As Bell reported in 1964 [1], certain inequalities that correlation functions of a local realistic theory must obey can be violated by quantum mechanics. Bell used the singlet state to demonstrate this. Likewise, a certain set of correlation functions produced by quantum measurements of a single quantum state can contradict local realistic theories. Since Bell's work, local realistic theories have been researched extensively [2, 3, 4]. Numerous experiments have shown that Bell inequalities and local realistic theories are violated [5, 6, 7].

In 1982, Fine presented [8] the following example: A set of correlation functions can be described with the property that they are reproducible by local realistic theories for a system in two-partite states if and only if the set of correlation functions satisfies the complete set of (two-setting) Bell inequalities. This is generalized to a system described by multipartite [9, 10] states in the case where two dichotomic observables are measured per site. We have, therefore, obtained the necessary and sufficient condition for a set of correlation functions to be reproducible by local realistic theories in the specific case mentioned above.

However, it was shown that such a "two-setting" local realistic model is disqualified if one imposes rotational invariance on local realistic models with respect to a measurement plane, where one has more than three spins [11]. Moreover, in a mixture of six-qubit Greenberger-Horne-Zeilinger (GHZ) states [12, 13], a generalized Bell inequality, which is derived under the assumption that there exist rotationally invariant local realistic models, disproves such a "two-setting" local realistic model stronger than a generalized Bell inequality, which is derived under the assumption that there exist local realistic models that are rotationally invariant with respect to a measurement plane [14].

Rotational invariance states that the value of the correlation function cannot depend on the local coordinate systems used by the observers. Therefore, we see that such a "two-setting" local realistic model depends on the local coordinate systems used by the observers in some situation. It was discussed [15] that there is a division among the measurement settings, those that admit local realistic models that are rotationally invariant with respect to a plane, and those that do not. This is another manifestation of the underlying contextual nature of local realistic theories of quantum experiments.

Here, we shall show that such a "two-setting" local realistic model is disqualified even though one has only two spins if we impose rotational invariance on local realistic models. This phenomenon can occur when the system is in a mixed two-qubit state. We analyze the threshold visibility for two-particle interference to reveal the disqualification mentioned above. We found that the threshold visibility is 0.75, which is more stringent than the one ($2(2/\pi)^2 \sim 0.81$) reported in Ref. 11. The result implies that explicit "two-setting" local realistic models cannot, in general, have the property that they are rotationally invariant.

The importance of the result of this paper can be addressed in conjunction with the convenience to create two-particle interference. In contrast, it is difficult to create multi-particle GHZ-type interference. Hence, our result provides a method to disqualify a rotationally-invariant local realistic theory experimentally easier than previous discussions in Refs. 11 and 14.

II. OMNIDIRECTIONAL GENERALIZED BELL INEQUALITY

In this section, we shall briefly review the generalized Bell inequality presented in Ref. 14. Consider two spin-$\frac{1}{2}$ particles, each in a separate laboratory. Let us parameterize the local settings of the jth observer with a unit vector $\vec{n}_j$ with $j = 1, 2$. One can introduce the "Bell" correlation function, which is the average of the product of the local results:

$$E(\vec{n}_1, \vec{n}_2,) = \langle r_1(\vec{n}_1) r_2(\vec{n}_2)\rangle_{\rm avg}, \tag{1}$$

where $r_j(\vec{n}_j)$ is the local result, ± 1, which is obtained if the measurement direction is set at $\vec{n}_j$. If the correlation function admits a rotationally invariant tensor structure familiar from quantum mechanics, we can introduce the following form:

$$E(\vec{n}_1, \vec{n}_2) = \hat{T} \cdot (\vec{n}_1 \otimes \vec{n}_2), \tag{2}$$

where $\otimes$ denotes the tensor product, $\cdot$ the scalar product in $\mathrm{R}^{3\times 2}$, and $\hat{T}$ is the correlation tensor, the elements of which are given by

$$T_{i_1 i_2} \equiv E(\vec{x}_1^{(i_1)}, \vec{x}_2^{(i_2)}), \tag{3}$$

with $\vec{x}_j^{(i_j)}$ being a unit vector of the local coordinate system of the jth observer; $i_j = 1, 2, 3$ gives the full set of orthogonal vectors defining the local Cartesian coordinates. The components of the correlation tensor are experimentally accessible by measuring the correlation function at the directions given by the bases vectors in which the tensor is written. Suppose one knows the values of all 3^2 components of the correlation tensor, $T_{i_1 i_2}$. Then, with the help of the formula in Eq. (2), one can compute the value of the correlation function for all other possible sets of local settings.

We shall derive a necessary condition for the existence of a rotationally invariant local realistic model of the rotationally invariant correlation function in Eq. (2). A correlation function has a rotationally-invariant local realistic model if it can be written as

$$E_{LR}(\vec{n}_1, \vec{n}_2) = \int d\lambda \rho(\lambda) I^{(1)}(\vec{n}_1, \lambda) I^{(2)}(\vec{n}_2, \lambda), \tag{4}$$

where λ denotes some hidden variable, $\rho(\lambda)$ is its distribution, and $I^{(j)}(\vec{n}_j, \lambda)$ is the predetermined "hidden" result of the measurement of all the dichotomic observables parameterized by any direction of $\vec{n}_j$. One can write the observable (unit) vector $\vec{n}_j$ in a spherical coordinate system as

$$\vec{n}_j(\theta_j, \phi_j) = \sin\theta_j \cos\phi_j \vec{x}_j^{(1)} + \sin\theta_j \sin\phi_j \vec{x}_j^{(2)} + \cos\theta_j \vec{x}_j^{(3)}, \tag{5}$$

where $\vec{x}_j^{(1)}$, $\vec{x}_j^{(2)}$, and $\vec{x}_j^{(3)}$ are the Cartesian axes relative to which spherical angles are measured.

The scalar product of the rotationally invariant local realistic correlation function, E_{LR} given in Eq. (4), with the rotationally invariant correlation function, E given in Eq. (2), is bounded by a specific number that depends on $\hat{T}$. We use the decomposition in Eq. (5) and introduce the usual measure $d\Omega_j = \sin\theta_j d\theta_j d\phi_j$ for the system of the jth observer. It was proven [14] that

$$\begin{aligned} (E_{LR}, E) \; &= \int d\Omega_1 \int d\Omega_2 E_{LR}(\theta_1, \phi_1, \theta_2, \phi_2) \\ &\times \; E(\theta_1, \phi_1, \theta_2, \phi_2) \leq (2\pi)^2 T_{\max}, \end{aligned} \tag{6}$$

where $T_{\max}$ is the maximal possible value of the correlation tensor component, maximized over choices of all possible local settings:

$$T_{\max} = \max_{\theta_1, \phi_1, \theta_2, \phi_2} E(\theta_1, \phi_1, \theta_2, \phi_2). \tag{7}$$

On the other hand, we have

$$(E, E) = (4\pi/3)^2 \sum_{i_1, i_2 = 1}^{3} T_{i_1 i_2}^2. \tag{8}$$

Therefore, the necessary condition for the existence of a rotationally invariant local realistic model of rotationally invariant correlations that involve the entire range of settings reads

$$\max \sum_{i_1, i_2 = 1,2,3} T_{i_1 i_2}^2 \leq \left(\frac{3}{2}\right)^2 T_{\max}, \tag{9}$$

where the maximization is taken over all independent rotations of local coordinate systems (or equivalently over all possible measurement directions).

III. VIOLATION OF ROTATIONAL INVARIANCE OF LOCAL REALISTIC MODELS

Consider two-qubit states:

$$\rho_{a,b} = V|\psi\rangle\langle\psi| + (1-V)\rho_{\mathrm{noise}} \; (0 \leq V \leq 1), \tag{10}$$

where $|\psi\rangle$ is the singlet state as $|\psi\rangle = \frac{1}{\sqrt{2}}(|+^a; -^b\rangle - |-^a; +^b\rangle)$. $\rho_{\mathrm{noise}} = \frac{1}{4}I$ is the random noise admixture. The value of V can be interpreted as the reduction factor of the interferometric contrast observed in the two-particle correlation experiment. The states $|\pm^j\rangle$ are eigenstates of the z-component Pauli observable σ_z^k for the jth observer. Here, a and b are the labels of the parties (say Alice and Bob). One can show that if the observers limit their settings to $\vec{x}_j^{(1)} = \hat{x}_j$, $\vec{x}_j^{(2)} = \hat{y}_j$, and $\vec{x}_j^{(3)} = \hat{z}_j$, then one has

$$\begin{aligned} &T_{11} = T_{22} = T_{33} = -V, \\ &T_{12} = T_{21} = 0, \\ &T_{13} = T_{31} = 0, \\ &T_{23} = T_{32} = 0. \end{aligned} \tag{11}$$

Thus, the maximal possible component of the correlation tensor is equal to $T_{\max} = V$. It is easy to see that

$$\max \sum_{i_1, i_2 = 1,2,3} T_{i_1 i_2}^2 = 3V^2. \tag{12}$$

Hence, the generalized Bell inequality is violated if $V > \frac{3}{4}$.

On the other hand, the set of experimental correlation functions is described with the property that they are reproducible by "two-setting" local realistic theories. See

the following relations along with the arguments in Ref. 8;

$$\begin{aligned}
&|T_{11}-T_{12}+T_{21}+T_{22}| \le 2V \le 2,\\
&|T_{11}+T_{12}-T_{21}+T_{22}| \le 2V \le 2,\\
&|T_{11}+T_{12}+T_{21}-T_{22}| = 0 \le 2,\\
&|T_{11}-T_{12}-T_{21}-T_{22}| = 0 \le 2;
\end{aligned} \tag{13}$$

$$\begin{aligned}
&|T_{22}-T_{23}+T_{32}+T_{33}| \le 2V \le 2,\\
&|T_{22}+T_{23}-T_{32}+T_{33}| \le 2V \le 2,\\
&|T_{22}+T_{23}+T_{32}-T_{33}| = 0 \le 2,\\
&|T_{22}-T_{23}-T_{32}-T_{33}| = 0 \le 2;
\end{aligned} \tag{14}$$

$$\begin{aligned}
&|T_{11}-T_{13}+T_{31}+T_{33}| \le 2V \le 2,\\
&|T_{11}+T_{13}-T_{31}+T_{33}| \le 2V \le 2,\\
&|T_{11}+T_{13}+T_{31}-T_{33}| = 0 \le 2,\\
&|T_{11}-T_{13}-T_{31}-T_{33}| = 0 \le 2.
\end{aligned} \tag{15}$$

Therefore, we have

$$\int d\lambda \rho(\lambda) I^{(1)}(\vec{x}_1^{(i)}, \lambda) I^{(2)}(\vec{x}_2^{(i)}, \lambda) = -V, \tag{16}$$

for i = 1, 2, and 3, and

$$\int d\lambda \rho(\lambda) I^{(1)}(\vec{x}_1^{(i)}, \lambda) I^{(2)}(\vec{x}_2^{(j)}, \lambda) = 0, \tag{17}$$

for $i \neq j$.

Please note that the singlet state is $U_1 \otimes U_2$ invariant [16]. Here, U_j are unitary matrices and $U_1 = U_2$. Hence, for the state $\rho_{a,b}$, we have

$$U_1^\dagger \otimes U_2^\dagger \rho_{a,b} U_1 \otimes U_2 = \rho_{a,b}. \tag{18}$$

Therefore, one has "two-setting" local realistic models for values of the correlations for the entire range in space by using many unitary operations in the form $U_1 \otimes U_2$. That is, we have

$$\int d\lambda \rho(\lambda) I^{(1)}(U_1 \vec{x}_1^{(i)} U_1^\dagger, \lambda) I^{(1)}(U_2 \vec{x}_2^{(i)} U_2^\dagger, \lambda) = -V, \tag{19}$$

for i = 1, 2, and 3, and

$$\int d\lambda \rho(\lambda) I^{(1)}(U_1 \vec{x}_1^{(i)} U_1^\dagger, \lambda) I^{(1)}(U_2 \vec{x}_2^{(j)} U_2^\dagger, \lambda) = 0, \tag{20}$$

for $i \neq j$.

Please note that that these models in Eqs. (16,17) cannot be ruled out by any two-setting Bell inequality. The "two-setting" local realistic models in Eqs. (19,20) are in the following structure because we used only unitary operations in the form $U_1 \otimes U_2$, $(U_1 = U_2)$:

$$\int d\lambda \rho(\lambda) I^{(1)}(\vec{n}_1, \lambda) I^{(2)}(\vec{n}_2, \lambda) = \begin{cases} -V, & \vec{n}_1 = \vec{n}_2, \\ 0, & \vec{n}_1 \cdot \vec{n}_2 = 0. \end{cases} \tag{21}$$

Therefore, no two-setting Bell inequality can rule out the models in Eqs. (16,17) because only "two-setting" local realistic models made by only commuting pairs of observables have nonvanishing values. Nevertheless, despite the fact that there exist "two-setting" local realistic models for all directions in Eqs. (19,20), and every $U_1 \otimes U_2$), these models cannot construct rotationally invariant local realistic models, and they are ruled out if $V > \frac{3}{4}$.

Thus, the situation is such that for any value of V, one can construct a "two-setting" local realistic model for the values of the correlation functions for the settings chosen in the experiment (Eqs. 16,17). One wants to construct an "omnidirectional" local realistic model for the entire range by using "two-setting" local realistic models by using many unitary operations in the form $U_1 \otimes U_2$ and $U_1 = U_2$ in (Eqs. 19,20), but these "two-setting" models must be consistent with each other, if we want to construct truly "omnidirectional" local realistic models beyond the 2^2 settings to which each of them pertains. Our result clearly indicates that this is impossible for $V > \frac{3}{4}$. That is, "two-setting" models built to reconstruct the 2^2 data points, when compared with each other, must be inconsistent; therefore, they are invalidated. The "two-setting" models must contradict each other. In other words, the explicit models given in Refs. 8-10 work only for the specific set of settings in the given experiment, but cannot be rotationally invariant; therefore, they are ruled out in some situations.

IV. SUMMARY

In summary, we have shown that a "two-setting" local realistic model is disqualified, even though one has only two spins, if we impose rotational invariance on local realistic models. This phenomenon can occur when the system is in a mixed two-qubit state. We analyzed the threshold visibility for two-particle interference to reveal the disqualification mentioned above. We found that the threshold visibility was 0.75, which is more stringent than the one $(2(2/\pi)^2 \sim 0.81)$ reported in Ref. 11. The result implies that explicit "two-setting" local realistic models cannot have the property that they are rotationally invariant.

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